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Group-Theoretic Approach to the Conservation Laws of the KP Equation in Lagrangian and Hamiltonian Formalism

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The conservation laws—precisely speaking, the basis of the conservation laws—are obtained through the use of Noether's theorem, Lie symmetry, and a theorem due to Ibragimov. Though in principle for each generator of Lie symmetry there should be a different conserved vector, due to the closed Lie algebra generated by the generators, some of these vectors become no longer independent. The theorem of Ibragimov is used to construct a basis in the case of the KP equation in three dimensions. It is then shown how the same analysis can be performed through the Hamiltonian formalism.

1. INTRODUCTION

From the beginning of the study of nonlinear equations, Lie symmetry has played a dominant role in analyzing their properties (Ames, 1972). Lie point symmetry has been especially used for obtaining the similarity solutions and conservation laws (Rund and Lovelock, 1975). Actually, for equations derivable from a variational principle the Noether theorem yields a link between the symmetry generators and conservation laws. On the other hand, symmetry generators always obey a closed Lie algebra. It was pointed out by Ibragimov (1969) that due to this underlying algebraic structure of the symmetry generators, the Noether theorem does not always lead to independent conservation laws. In the following we will discuss the basic theorem of Obragimov and apply it to study the group-theoretic properties of the conservation laws associated with the KP equation. Though the Lie

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symmetry of KP was obtained previously and its relation to Kac–Moody algebra discussed in (Winternitz *et al.*, 1985), the group-theoretic study of the conservation laws has yet to be done. Some properties of these conservation laws were given by Infeld (1981).

2. FORMALISM

Before proceeding to the actual calculation, let us review the basic formalism laid down in Ibragimov (1969). Suppose the evolution equation under consideration reads

$$u_t = f(u, x, t, u_x, u_{xx}, \ldots)$$

Then the Lie–Backlung operator is written as

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \sum \mathcal{G}_{i_1, i_2, \ldots, i_k} \frac{\partial}{\partial u_{i_1, i_2, \ldots, i_k}} \tag{1}$$

where u_i denotes $\partial u/\partial x_i$ and the $\mathcal{G}_{ijk\ldots}$ are computed according to

$$\mathcal{G}_{ijkl\ldots} = D_l D_j \cdots D_n(\eta - \xi^i u_i) + \xi^\sigma u_{\sigma ijkl} \tag{2}$$

In these equations D_i denotes total derivative with respect to the ith coordinate,

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ii_1} \frac{\partial}{\partial u_{i_1}} + \cdots \tag{3}$$

The Noether operator is then constructed as follows:

$$\begin{aligned} N^i = \xi^i + (\eta - \xi^i u_i) \Bigg\{ & \frac{\partial}{\partial u_i} + \sum_{s \geq 1} (-1)^s D_{j_1 \ldots j_s} \frac{\partial}{\partial u_{i, j_1 \ldots j_s}} \\ & + \sum_{r \geq 1} D_{k_1} D_{k_2} \cdots D_{k_r} (\eta - \xi^l u_l) \Bigg[\frac{\partial}{\partial u_{ik_1 \ldots k_r}} \\ & + \sum_{s \geq 1} (-1)^s D_{j_1 \ldots j_s} \frac{\partial}{\partial u_{i, k_1 \ldots k_r, j_1 \ldots j_s}} \Bigg] \Bigg\} \end{aligned} \tag{4}$$

The conserved vector is then obtained by operating with N^i on $\mathcal{L}$, that is,

$$C^i = N^i(\mathcal{L}) \tag{5}$$

is a conserved vector, satisfying $D_i(C^i) = 0$. Let us now suppose that the different generators of Lie symmetry pertaining to a particular nonlinear equation are X_i, $i = 1, \ldots, n$, and they generate a Lie algebra 'L'. If we denote the conservation laws corresponding to each X_i as C^i, then we can state the basic assertion of Ibragimov as follows:

If, from the Lie algebra 'L' we have the relation

$$[X_i, X_j] = C_k \tag{6}$$

then the conserved vectors C^i and C^k are equivalent in the sense that

$$\bar{X}_i(C_j) = C_k + \text{total derivative terms} \tag{7}$$

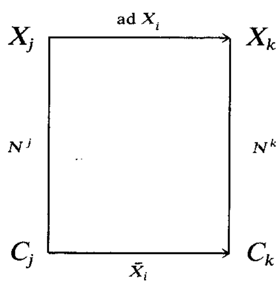
where $\bar{X}_i$ is an operator constructed from X_i according to

$$\begin{aligned} \bar{X}_i &= X_i - \xi^j D_j = \bar{\eta}^\alpha \, \partial/\partial u^\alpha + \cdots \\ \bar{\eta}^\alpha &= \eta^\alpha - \xi^j u_j^\alpha \end{aligned} \tag{8}$$

If equation (6) is written as

$$\text{ad } X_i(X_j) = X_k \tag{9}$$

then the following commutative diagram must be true:



Thus, the above theorem dictates that the two conservation laws C_k and C_j are not independent. In the following we will use this theorem repeatedly in the case of the KP equation.

3. THE KP EQUATION

The KP equation is written in potential form as

$$(u_t + \tfrac{3}{4}u_x^2 + \tfrac{1}{4}u_{xxx})_x + \tfrac{3}{4}\sigma u_{yy} = 0 \tag{10}$$

which can be generated from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}u_t u_x + \tfrac{1}{4}u_x^3 - \tfrac{1}{8}u_{xx}^2 + \tfrac{3}{8}\sigma u_y^2 \tag{11}$$

The symmetry generators of (10) are

$$
\begin{aligned}
X_A &= t^3\frac{\partial}{\partial t}+(xt^2-\tfrac{4}{3}\sigma y^2 t)\frac{\partial}{\partial x}+2yt^2\frac{\partial}{\partial y}\\
&\quad+(\tfrac{2}{3}x^2t-2ut^2-\tfrac{8}{9}\sigma y^2 x)\frac{\partial}{\partial u}\\
X_B &= t^2\frac{\partial}{\partial t}\left(\frac{2}{3}xt-\frac{y}{9}\sigma y^2\right)\frac{\partial}{\partial x}+(\tfrac{2}{9}x^2-\tfrac{4}{3}ut)\frac{\partial}{\partial u}+\tfrac{4}{3}yt\frac{\partial}{\partial y}\\
X_C &= t\frac{\partial}{\partial t}+\frac{1}{3}x\frac{\partial}{\partial x}+\frac{2}{3}y\frac{\partial}{\partial y}-\frac{2}{3}u\frac{\partial}{\partial u}\\
X_D &= \frac{\partial}{\partial t}\\
X_E &= -\frac{4}{3}\sigma yt\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial y}-\frac{8}{9}\sigma yx\frac{\partial}{\partial u}\\
X_k &= -\frac{2}{3}\sigma y\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}\\
X_L &= \frac{\partial}{\partial y},\qquad X_M = t\frac{\partial}{\partial x}+\frac{2}{3}x\frac{\partial}{\partial u},\qquad X_N=\frac{\partial}{\partial x},\qquad X_P=\frac{\partial}{\partial u}
\end{aligned}
\tag{12}
$$

The commutation rules and the Lie algebra generated by these are given in Table I.

Now, following Infeld (1981), we construct the (x, y, t) components of the Noether operators and apply them to $\mathscr{L}$ to get the (x, y, t) components of the conserved vectors C. Here we use the following notation:

$$
\begin{aligned}
N_\alpha^i &= i\text{th component } (i = x, y, \text{ or } t)\\
&\quad \text{corresponding to the operators } X_\alpha
\end{aligned}
\tag{13}
$$

Then we get

$$
\begin{aligned}
N_N^x(\mathscr{L}) &= C_N^x = -\tfrac{1}{2}u_x^3+\tfrac{1}{8}u_{xx}^2-\tfrac{1}{4}u_x u_{xxx}+\tfrac{3}{8}\rho u_y^2\\
N_N^y(\mathscr{L}) &= C_N^y = -\tfrac{3}{4}\sigma u_x u_y\\
N_N^t(\mathscr{L}) &= C_N^t = -\tfrac{1}{2}u_x^2
\end{aligned}
\tag{14}
$$

Similar computations can also be performed for other generators. But we now try to visualize the group properties with the help of Table I.

Note that

$$
[X_M, X_N] = \tfrac{2}{3}X_0 \qquad [X_D, X_E] = -2X_K \tag{15}
$$

Table I. Commutation Rules for the Generator[a]

	A	B	C	D	E	K	L	M	N	P
A	0	X_{AB}	X_{AC}	$-3B$	X_{AE}	X_{AK}	$-2E$	$\frac{2}{3}xt^2p$	X_{AN}	$2t^2p$
B		0	$-B-\frac{2}{3}$	$-2C$	X_{BE}	$-\frac{1}{3}E$	$-\frac{4}{3}E$	X_{BM}	$-\frac{2}{3}M$	$\frac{4}{3}tp$
C			0	$-C$	X_{CE}	$\frac{1}{3}K$	$-\frac{1}{3}L$	$M-\frac{1}{3}tN$	$-\frac{1}{3}N$	$\frac{2}{3}p$
D				O	X_{DE}	L	0	N	0	0
E					0	X_{EK}	$\frac{4}{3}\sigma M$	0	$\frac{8}{9}\rho yp$	0
K							$\frac{2}{3}\sigma N$	$-\frac{4}{3}\sigma yp$	0	0
L					0		0	0	0	0
M								0	$\frac{2}{3}p$	0
N									0	0
P										0

[a]The Xs are given as follows:

$X_{AB}=\frac{4}{3}tX_A-\frac{8}{9}\sigma^2y^2yX_N+\frac{1}{3}t^4X_D$
$\quad+[-4\sigma^2y^4+(80/27)\sigma y^2xt+\frac{2}{3}x^2t^2]X_P$

$X_{AC}=-2X_A+(8/27)\sigma y^2xX_P-\frac{2}{9}x^2tX_P$

$X_{AK}=-tX_E+\frac{2}{3}\sigma yX_M+\frac{4}{9}\sigma y(3xt-\frac{4}{3}\sigma y)X_P$

$X_{AN}=2tX_M-\frac{4}{9}\sigma y(4x-t)X_P$

$X_{BN}=\frac{4}{3}tX_M-t^2X_N-(8/27)\sigma y^2X_P$

Applying the above theorem, we can say (TDT = total derivative terms)

$$\bar{X}_M(C_N)=\frac{2}{3}C_p+\text{TDT}$$
$$\bar{X}_D(C_E)=-2C_K+\text{TDT} \tag{16}$$

from which we infer that C_P and C_K can be constructed from C_N and C_E. We also have

$$[X_D, X_K]=X_L$$

so that C_L can be constructed by operating with $\bar{X}_D$ on C_K, which is known from equation (8). That is,

$$\bar{X}_D(C_K)=C_L \tag{17}$$

Thus, from the knowledge of C_N and C_E we can manufacture C_P, C_K and C_L.

Next let us consider the commutation relations with X_C, which have the following implications according to the above theorem:

$$[X_C, X_A]=2X_A-\tfrac{8}{27}\sigma y^2xX_P+\tfrac{2}{9}x^2tX_P \tag{18a}$$

implies $\Rightarrow$ $\bar{X}_A(C_C)=2C_A-(\frac{8}{27}\sigma y^2x-\frac{2}{9}x^2t)X_P$

$$[X_C, X_B]=X_B+\tfrac{2}{37}x^2X_P \tag{18b}$$

implies $\Rightarrow$ $\bar{X}_B(C_C) = C_B + \frac{2}{27}x^2 C_P$

$$[X_C, X_D] = -X_D \tag{18c}$$

implies $\Rightarrow$ $\bar{X}_D(C_C) = -C_D$

$$[X_C, X_E] = \tfrac{4}{3}X_E - \tfrac{8}{27}\sigma yx X_P \tag{18d}$$

implies $\Rightarrow$ $\bar{X}_E(C_C) = \frac{4}{3}C_E - \frac{8}{27}\sigma yx C_P$

$$[X_C, X_K] = \tfrac{1}{3}X_K \tag{18e}$$

implies $\Rightarrow$ $\bar{X}_K(C_C) = \frac{1}{2}C_K$

$$[X_C, X_L] = -\tfrac{2}{3}X_L \tag{18f}$$

implies $\Rightarrow$ $\bar{X}_L(C_C) = -\frac{2}{3}C_L$

$$[X_C, X_M] = X_M - \tfrac{1}{3}tX_N \tag{18g}$$

implies $\Rightarrow$ $\bar{X}_M(C_C) = C_M - \frac{1}{3}tC_N$

$$[X_C, X_N] = -\tfrac{1}{3}X_N \tag{18h}$$

implies $\Rightarrow$ $\bar{X}_N(C_C) = -\frac{1}{3}C_N$

$$[X_C, X_P] = \tfrac{2}{3}X_P \tag{18i}$$

implies $\Rightarrow$ $\bar{X}_P(C_C) = \frac{2}{3}C_P$

Thus, if we know X_C and the corresponding current C_C, all the other currents can be constructed via the Ibragimov theorem and the operators $\bar{X}_i$ corresponding to $i = L, M, N, P, E, A, B, D$, and K. So we can positively conclude that $\{C_C\}$ is the basis of all the conservation laws.

4. COMPUTATIONS AND OTHER INFERENCES

In the above analysis we have obtained the basis of the conservation laws by the use of a theorem due to Ibragimov for the KP equation. The analysis indicates that it is sufficient to use Noether's theorem only once to calculate the conservation law in the case of the generator X_C and obtain the other laws via the operator $\bar{X}_i$. Here we analyze by direct calculation the other consequences of the theorem in relation to the group-theoretic properties of the conservation laws.

(i) First let us consider three operators X_a, X_b, and C_c such that $[X_b, X_c] = 0$; then it follows from the above that

$$C_{acb} = C_{abc} \quad \text{with} \quad C^i_{abc} = \bar{X}_c\bar{X}_b(C^i_a) \tag{19}$$

We can verify this very easily, but to simplify the calculation, let us take a particular example. From the commutation table we have

$$[X_L, X_M] = 0 \tag{20}$$

So we consider

$$
\begin{aligned}
C^3_{KML} &= \bar{X}_L(C^3_{KM}) \\
&= -u_y \frac{\partial}{\partial u}(\tfrac{4}{9}\sigma y u_x - \tfrac{2}{3}\sigma y t u_x u_{xx} \\
&\quad -\tfrac{1}{3}tu_y - \tfrac{1}{2}t^2 u_y u_{xx} + \tfrac{1}{2}t^2 u^x u_{xy} \\
&= -\tfrac{4}{9}\sigma y u_{xy} + \tfrac{2}{3}\sigma y t u_{xx} u_{xy} + \tfrac{2}{3}\sigma y t u_x u_{yxx} \\
&\quad +\tfrac{1}{3}tu_{yy} + \tfrac{1}{2}t^2 u_{yy} - \tfrac{1}{2}t^2 u_y u_{yxx} - \tfrac{1}{2}t^2 u_{yx} u_{yx} - \tfrac{1}{2}t^2 u_x u_{xyy}
\end{aligned}
\tag{21}
$$

On the other hand,

$$
\begin{aligned}
C^3_{KLM} &= \bar{X}_M(C^3_{KL}) \\
&= \tfrac{2}{3}(x - tu_x)\frac{\partial}{\partial u}(\tfrac{2}{3}\sigma y u_x u_{xy} + \tfrac{1}{2}tu_y u_{xy} + \tfrac{1}{2}tu_x u_{yy}) \\
&= \tfrac{2}{3}\sigma y u_{xy}(\tfrac{2}{3} - tu_{xx}) + \tfrac{2}{3}\sigma y u_x(-tu_{xxy}) \\
&\quad +\tfrac{1}{2}tu_u(-tu_{xxy}) + \tfrac{1}{2}tu_{xy}(-tu_{xy}) \\
&\quad +\tfrac{1}{2}tu_{yy}(\tfrac{2}{3} - tu_{xx}) + \tfrac{1}{2}tu_x(-tu_{xyy})
\end{aligned}
\tag{22}
$$

which can be seen to be equal to the expressions in (21).

(ii) Next let us consider two generators that do not commute with each other. Say we have two generators X_α and X_β such that $[X_\alpha, X_\beta] = \lambda X_\gamma$; then from the Ibragimov theorem it follows that

$$
C^i_{\gamma\alpha\beta} - C^i_{\gamma\beta\alpha} = \lambda \bar{X}_\beta(C^i_\gamma)
\tag{23}
$$

For example, from Table I we have $[C, N] = \frac{1}{3}N$ and from the basic conservation law C_C we can construct C_M. In this computation we only consider the third component of C_M. It is not difficult to see that

$$
C^3_M = \tfrac{1}{3}xu_x - \tfrac{1}{2}tu_x^2
\tag{24}
$$

So

$$
\begin{aligned}
C^3_{MC} = \bar{X}_C(C^3_M) &= \tfrac{1}{3}x(-u_x - tu_{tx} - \tfrac{1}{3}xu_{xx} - \tfrac{2}{3}yu_{xy}) \\
&\quad + tu_x(u_x + tu_{tx} + \tfrac{1}{3}xu_{xx} + \tfrac{2}{3}yu_{xy})
\end{aligned}
$$

Hence

$$
\begin{aligned}
C^3_{MCN} = \bar{X}_N(C^3_{MC}) &= \tfrac{1}{3}xu_{xx} + \tfrac{1}{3}txu_{xxt} \\
&\quad + \tfrac{1}{9}x^2 u_{xxx} + \tfrac{2}{9}xyu_{xxy} - 2tu_x u_{xx} - t^2 u_{tx} u_{xx} - t^2 u_x u_{xxt} \\
&\quad - \tfrac{1}{3}xtu_{xx}^2 - \tfrac{1}{3}xty_x u_{xxx} - \tfrac{2}{3}ytu_{xy}u_{xx} - \tfrac{2}{3}ytu_x u_{xxy}
\end{aligned}
\tag{25}
$$

On the other hand, we have

$$C^3_{MN} = \bar{X}_M(C^3) = -\tfrac{1}{3}xu_{xx} + tu_xu_{xx}$$

$$\begin{aligned} C^3_{MNC} = \bar{X}_C(C^3_{MN}) &= \tfrac{4}{9}xu_{xx} + \tfrac{1}{3}xtu_{txx} + \tfrac{1}{9}x^2u_{xxx} \\ &\quad + \tfrac{2}{9}xyu_{xxy} - tu_xu_{xx} - t^2u_xu_{tx} - \tfrac{1}{3}xtu^2_{xx} \\ &\quad - \tfrac{2}{3}ytu_{xx}u_{yx} - \tfrac{4}{3}tu_xu_{xx} - t^2u_xu_{txx} \end{aligned} \tag{26}$$

Thus we have

$$\begin{aligned} C^3_{MCN} - C^3_{MNC} &= -\tfrac{1}{9}xu_{xx} + \tfrac{1}{3}tu_xu_{xx} \\ &= \tfrac{1}{3}\bar{X}_N(C^3_M) \end{aligned} \tag{27}$$

The main idea of these computations is that repeated application of X_i does not always generate new conservation laws.

(iii) Finally, it can be inferred that only those conservation laws are independent that are generated from two operators that commute between each other. In the present case, since no generator commutes with X_C, we have no such set in this case.

5. THE HAMILTONIAN APPROACH

Though we have deduced our main results, here we indicate another approach in which some of the above results can also be derived. In the Hamiltonian framework the KP equation is obtained as

$$\delta U_x/\delta t = -\delta H/\delta u$$

where

$$H = \tfrac{1}{4}u_x^3 + \tfrac{3}{8}\sigma u_y^2 - \tfrac{1}{6}u_{xx}^2 \tag{28}$$

We will use here an elegant formalism of Olver, in which he showed how one can manufacture integral invariants (the time component of the conserved vector) through the use of the basic two-form Ω and the Hamiltonian H.

The basic two-form Ω is written as

$$\Omega = du_x \wedge du \tag{29}$$

We then evaluate the inner product of the vector field X_α with Ω. For example,

$$\begin{aligned} C^t_{N'} &= \omega_{N'} = X_{N'} \lrcorner\, \Omega = (-u_x\,\partial u) \lrcorner\, du_x \wedge du = d^*(-\tfrac{1}{2}u_x^2) \\ C^t_{M'} &= \omega_{M'} = X_{M'} \lrcorner\, \Omega = d^*(-\tfrac{1}{2}tu_x^2 - \tfrac{2}{3}u) \\ C^t_{L'} &= \omega_{L'} = X_{L'} \lrcorner\, \Omega = d^*(-\tfrac{1}{2}u_xu_y) \\ C^t_{E'} &= \omega_{E'} = X_{E'} \lrcorner\, \Omega = d^*(\tfrac{2}{3}\sigma ytu_x^2 - \tfrac{1}{2}t^2u_xu_y + \tfrac{8}{9}\sigma yu) \\ &\vdots \end{aligned} \tag{30}$$

Now, to derive the classification scheme, we consider the different situations when (i) $[X_E, X_M]=0$ and (ii) $[X_E, X_M]\neq 0$.

(i) When $[X_E, X_M]=0$, we get

$$\omega_{E'}(\omega_{M'} \lrcorner \Omega)=\omega_{M'} \lrcorner \omega_{E'}(\Omega)+[\omega_{E'}, \omega_{M'}] \lrcorner \Omega \tag{31}$$

Now, by definition, the basic two-form Ω is an integral invariant and hence $X_\alpha \lrcorner \Omega$ is another integral invariant.

(ii) For the case $[X_E, X_M]\neq 0$ we get, for example, for X_C and X_M

$$[X_C, X_M]=X_M-\tfrac{1}{3}tX_N \tag{32}$$

We then get

$$\begin{aligned}\omega_{C'}(\omega_{M'} \lrcorner \Omega)&=\omega_{M'}(\omega_{C'} \lrcorner \Omega)+[\omega_{C'}, \omega_{M'}] \lrcorner \Omega\\ &=\omega_{M'}-\tfrac{1}{3}t\omega_{N'}\end{aligned}$$

Similarly, for $[X_E, X_L]=\frac{4}{3}\sigma X_M$ we get

$$\begin{aligned}\omega_{E'}(\omega_{L'} \lrcorner \Omega)&=\omega_{L'}(\omega_{E'} \lrcorner \Omega)+[\omega_{E'}, \omega_{L'}] \lrcorner \Omega\\ &=\tfrac{4}{3}\sigma\omega_{M'}\end{aligned} \tag{33}$$

So the new integral invariants will contain a part of the old ones, and, as before, the dependence of the conserved quantitites can be ascertained.

6. DISCUSSION

In the above analysis we have obtained the minimal set of conserved vectors for the KP equation. The analysis also indicates how and when one can expect to generate independent conserved quantities.

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